Instabilities and propagation properties in a fourth-order reaction–diffusion equation

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Abstract

In this paper we investigate the instability and the propagation properties of a class of reaction–diffusion equations of fourth order. Two examples are introduced, the extended Fisher–Kolmogorov equation (EFK), and the Swift–Hohenberg equation (SH). Both have been studied before by related methods (see for example, Peletier and Rottschafer, 2004 [19]; Van Saarloos, 2003 [24]) but the analysis here will support the introduced linear mechanism in front selection. These two equations support a patterned front solutions, and the double eigenvalue mechanism is used to provide evidence for that and to determine a minimal front speed.

1. Introduction

Whenever the spatial spread of a population or chemical species is of importance, reaction–diffusion equations are used. For spatial spread, reaction–diffusion models have successfully been used in epidemic problems, pattern formation in different biological and ecological systems and in signal transport. Good overviews are given in Britton [5] and Grindrod [11]. From a theoretical point of view, one may distinguish two types of reaction–diffusion structures: (i) global structures resulting from intrinsic symmetry-breaking instabilities, e.g., Turing structures [21], and (ii) localized structures associated with fronts, i.e., steep spatial changes of concentration or densities which correspond to transitions between two states with fast kinetics, e.g., traveling waves [5]. In many natural phenomena we encounter propagating fronts separating different phases. Propagating fronts play an important role in the spread of epidemics, in population dynamics, or the propagation of flames and chemical reactions. Therefore, reaction–diffusion equations have become a prototype for describing propagating front behavior, form chemical waves to biological population.

1.1. Traveling waves and front propagation

A traveling wave is a wave which travels at constant speed without change in shape. If \( u(x, t) \) represents a traveling wave, the shape of \( u \) will be the same for all time and the speed of propagation of this shape is a constant. If we look at this wave in a traveling frame moving at the same speed it will appear stationary [16]. One of the most important properties of nonlinear parabolic systems is their ability to support traveling wave solutions. Unlike the linear wave equation, for example, which is hyperbolic and propagates any wave profile with a specific speed, reaction–diffusion equations may allow various wave profiles to propagate, each one with its own characteristic speed [11].

Traveling wave solution can be written in the form \( u(x, t) = V(z) = V(x - ct) \) for some velocity \( c \). Plane wave is a class of traveling waves with \( V(z) = U(z \cdot s) \) for some vector \( s \) (i.e., \( u = U(z \cdot s - ct) \), \( c \) a scalar). This class of waves, plane waves, is...
categorized in one dimension as [9]: (1) wave trains ($U$ periodic), (2) fronts ($U(\infty)$ and $U(\infty)$ exist and are unequal) and (3) pulses ($U(\pm \infty)$ exist and are equal; $U$ not constant). There are other forms in two dimension ($x = \{x, y\}$, $x = r \cos \theta$, $y = r \sin \theta$), such as: Target patterns ($u(x, t) = U(r, t)$, $U$ periodic in $t$), and Rotating spiral patterns ($u(x, t) = U(r, \theta - ct)$, $U$ periodic in second argument).

The propagation of a front into an unstable state is a problem that emerges in many branches of the natural sciences. These fronts may be classified as: (1) Uniformly translating fronts, which are in the form $u(z) = u(x - ct)$, where $c$ is the front speed. In this class of fronts invasion could be either monotonic or oscillatory (see Fig. 1(a) and (b), which represent possible solutions of Fisher’s equation (1)). (2) Pattern forming fronts, a front that generates a nontrivial pattern behind the wavefront. The front has a finite speed while the pattern is often stationary (see Fig. 1(c), represent possible solution of SH equation [6]). Thus these pattern fronts are typically not in the form $u(x - ct)$, and instead they are spatially and temporally periodic: they are of the type $u(z, t) = u(x - ct, t)$, with $u(z, t)$ periodic in $t$ with period $T$. $u(z, t) = u(z, t + T)$, thus in our analysis the perturbations are assumed in that form as we will see later.

1.2. Front selection

The prototypical model for reaction–diffusion systems is the Fisher-type nonlinear diffusion equation (scalar monostable), which we use here to illustrate some general principles:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + F(u),$$

where $u > 0$ may be interpreted as a population density, $F(0) = F(1) = 0$. This equation was introduced in 1937 by Fisher [10], with $F(u) = u(1 - u)$. At the same time by Kolmogorov, together with Petrovskii and Piskunov [13] (hereafter (1) referred to as FKPP). In their work of 1937, Kolmogorov et al. proved the existence of front solutions $u = U(x - ct)$, characterized by their velocity, $c$, such that

$$c \geq c_0 = 2\sqrt{F(0)}$$

and this result is obtained by a linearization about $u = 0$. Moreover, under some assumption on $F$, they proved that the FKPP-equation, Eq. (1), with a sufficiently decaying initial data has solutions with speed $c_0$. For more general monostable equations, it was shown rigorously by Aronson and Weinberger [2], for a sufficiently localized initial condition the solutions of (1) evolve into fronts with a minimal allowed speed $c_{\text{min}}$, such that

$$2\sqrt{F(0)} \leq c_{\text{min}} \leq 2\sup_u \sqrt{F(u)/u},$$

thus the propagating speed is either equal to or larger than $c_0$. Also, they showed that a monotonic traveling wave exists for all speeds $c \geq c_{\text{min}}$, and none for $c < c_{\text{min}}$. Therefore, from these results two selection mechanisms appeared: a linear and nonlinear selection of the propagation speed. In a linear selection mechanism the front dynamics can be understood by linear analysis since it is essentially determined by linearization near the unstable steady state ($u = 0$ in case of FKPP equation), so the front is pulled by its leading edge (see Fig. 1(a)), and in this case the selected front is called pulled front. However, for the selected fronts with speeds larger than the linear front speed, the details of the nonlinearity of the reaction term, $F(u)$, are important. In this case, the front dynamics are referred to as pushed, meaning that the front is pushed by its (nonlinear) interior, and a nonlinear analysis is required to determine the front speed. A nonlinear selection principle has been proposed to that aim (see [24]). Fisher’s equation has been studied extensively, considering the traveling wave existence.

Fig. 1. Schematic representation of some front types, all moving to the right with speed $c$. (a) Monotonic uniform translating front, represents solution of Fisher’s equation (1) when speed $c > 2$. (b) Front invading the unstable state in oscillatory manner, represents solution of Fisher’s equation (1) when speed $c < 2$. (c) Pattern forming front, a front moving to the right leaving a pattern behind. There are possible states behind the front, such as limit cycles, stationary patterns, oscillatory patterns, and spatio-temporal patterns.
problem and the speed of propagation. There is now a great deal of literature on this subject. A few scenarios have been proposed regarding the selection mechanism on some nonlinear reaction–diffusion equations, many for scalar equations. Some of the famous methods used are the marginal stability hypothesis (linear selection) \[6,3,22,8\] (for a review see [23]), structural stability hypothesis \[17\], construction of exact solutions \[7\], variational methods \[4,12\], and asymptotic methods (see for example Leach \[14\]).

2. Double root mechanism and the FKPP equation

In this section a linear speed selection mechanism is introduced. The mechanism gives some insights into the selected speed of invasion of an unstable state by a stable one, as described by fixed form of traveling wave and by a modulated traveling wave.

To illustrate some general principles in the selection mechanism, we use Eq. (1), with \( F = u(1 - u) \),

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1 - u),
\]

(4)

which has two steady states: \( u = 1 \) which is a stable steady state, while \( u = 0 \) is unstable, the two states are spatially homogeneous. If we look for a traveling wavefront solution to (4) for which \( 0 \leq u \leq 1 \), oscillatory invasion is not accepted, the allowable wave profile must satisfy the monotonicity of the solution (see figure see Fig. 1(a)). The minimum wave speed is \( c_{\text{min}} = c_0 = 2 \), and this speed can be obtained directly from (2). This minimum speed is exactly the wave speed at which the two eigenvalues associated with the leading edge of the front switch from complex to real (the profile can be one of the two profiles displayed in Fig. 1(a) and (b)). Thus, we can say that this wave speed occurs when the characteristic equations has a double root. The characteristic equation is obtained when we substitute the ansatz \( u = e^{\mu t} \) (fixed form traveling wave) into the linearized form of Eq. (4) in the traveling wave coordinates \( (u(x, t) = u(z), z = x - ct) \) around the unstable state \( u = 0 \). The above results can be found in many articles and text books, and discussed in details using the phase plane method in [15].

Now we aim to demonstrate how we can use this double root criterion to determine the linear (pulled) regime, but with a modulated traveling wave solution is assumed. This modulated form enables us to discuss the existence of a patterned front. In the analysis, we introduce the traveling wave coordinates \( (z, t) = (x - ct, t) \) into differential equations, then we linearize the obtained equation around the unstable rest state. This results in a perturbation equation. After that we substitute the ansatz (corresponding for \( \nu \neq 0 \) to modulated traveling wave solution with a time period \( T = 2\pi/\nu \))

\[
u = e^{\mu t} e^{\iota z},
\]

(5)

into the perturbation equation to obtain a characteristic equation

\[
Q(\mu; c, \nu) = 0,
\]

(6)

where \( Q \) is a polynomial in the eigenvalue \( \mu \) (its degree equals the order of the differential equation). The traveling front parameters are \( \nu \) (modulating frequency) and \( c \) (wave speed), which are both real. From (5), when the modulating frequency \( \nu \) is zero, the front may be a uniformly translating one, \( u(x - ct) \), or a modulated one, but in the latter \( T \) (the time period) cannot be determined by the liberalization. However if the modulating frequency \( \nu \) takes a nonzero value the wave is necessarily of modulated type with \( e^{\iota \nu t} = 1 \), so that \( T = 2\pi/\nu \) or some integer multiple of that value. If a steady state periodic pattern is left behind the wavefront it will typically have a spatial wave length \( 2\pi c/\nu = ct \).

A linear front speed is the speed at which a double eigenvalue occurs. Thus we need to determine the wave speed that satisfies the characteristic equation (6) and the double root equation

\[
\frac{\partial}{\partial \mu} Q(\mu; c, \nu) = 0.
\]

(7)

Now let us apply the above analysis on the FKKP-Eq. (4). In the traveling wave coordinates \( (z, t) = (x - ct, t) \), linearizing around the unstable state \( u = 0 \), and then substituting the perturbation displayed in (5), results in the characteristic equation (6) where

\[
Q(\mu; c, \nu) = \mu^2 + c\mu + 1 - i\nu
\]

(8)

and a minimal front speed \( c = c_0 \) is the speed that satisfies (6) and (7). Therefore, \( c_0 \) is given by

\[
\mu^2 + c_0\mu + 1 - i\nu = 0, \quad 2\mu^2 + c_0 = 0
\]

(9)

and when we solve, we find that \( c_0 = 2, \mu = -1 \) and \( \nu = 0 \). This suggests that for all wave speeds \( c \geq c_0 = 2 \), a monotonic front solution exists (the stable state \( u = 1 \) invades \( u = 0 \) in monotonic manner) and a patterned front solution \( (\nu \neq 0) \) does not occur.

We have demonstrated the propagation properties in the FKPP-equation and determined the minimum front speed using the double root mechanism with a modulated traveling wave solution. The task was easy as the eigenvalues are known
explicitly. This in general cannot happen in higher order equations, thus we need more computations. Therefore, in the following two higher order differential equations are discussed.

3. Higher order scalar reaction–diffusion equation

In this section we investigate the instability and the propagation properties of reaction–diffusion equations of fourth order. We study two equations, the extended Fisher Kolmogorov equation (EFK), and the Swift–Hohenberg equation (SH). Both have been studied before by related methods (see, [23,4,20,18,19]) but the analysis here will support the introduced linear mechanism in front election. These two equations support a patterned front solutions, and in this chapter the double eigenvalue mechanism is used to provide evidence for that and to determine a minimal front speed. In our discussion for each equation, we start with linear stability analysis, then we perform a traveling wave analysis to uncover the propagation properties of the front solution. A minimal front speed is determined, also indicating the type of the front (patterned front ($v \neq 0$) or not ($v = 0$) if we adopt the simplest assumptions about the form of the front that are consistent with the linearized analysis).

3.1. The extended Fisher–Kolmogorov equation

The EFK equation is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \gamma \frac{\partial^4 u}{\partial x^4} + u - u^3 \quad (\gamma > 0). \quad (10)$$

Assume that $u = u_s$ is the rest state, which can be either 0 or ±1. Linearizing (10) about the steady state gives the perturbation equation

$$\frac{\partial \tilde{u}}{\partial t} = \frac{\partial^2 \tilde{u}}{\partial x^2} - \gamma \frac{\partial^4 \tilde{u}}{\partial x^4} + (1 - 3u_s^2) \tilde{u} \quad (\gamma > 0), \quad (11)$$

where $\tilde{u}$ is the perturbation (we can drop hats for simplicity). We study the evolution modes of a given wave number $k$, where the perturbation $\tilde{u} = \tilde{u}_0 e^{ikx} e^{\sigma t}$, and $\sigma$ is the temporal growth rate. Substituting into (11) results in the dispersion relation

$$\sigma = -k^2 - \gamma k^4 + (1 - 3u_s^2). \quad (12)$$

The temporal growth rate $\sigma$ is plotted versus the wave number $k$, shown in Fig. 2. Thus we can say that the steady states $u_s = \pm 1$ are stable and $u_s = 0$ is unstable for the band $0 \leq k < k_+$, where $k_+ = (-1 + (1 + 4\gamma^{1/3})/2\gamma$. For this type of instability, while a non-trivial pattern can arise when the unstable state is perturbed, it is the uniform perturbation that grows most rapidly.

Now we aim to discuss the front solution properties of the EFK equation. Suppose that there are two rest states 0 and 1, where $u(\infty) = 0$ and $u(-\infty) = 1$. A front solution connecting these two states can exist, and in the following we determine the minimal front speed. Linearizing (10) around the unstable steady state $u = 0$, and in the traveling wave coordinates $(z,t) = (x - ct, t)$, one can deduce the linearized equation

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial z} = \frac{\partial^2 u}{\partial z^2} - \gamma \frac{\partial^4 u}{\partial z^4} + 1 \quad (13)$$

![Fig. 2. Dispersion relation of (10) and (12). The solid line represent the growth rate when the steady state $u = 0$, and the dashed when $u = \pm 1.$](image)
and substituting the ansatz $u = e^{it} e^{az}$ gives the characteristic equation
\[ Q = \gamma^4 - \mu^4 - c\mu - 1 + iv = 0, \tag{14} \]
where $v$ and $c$ are real and $\mu$ is complex. The minimal allowable wave speed $c = c_0$ occurs when there is a double root for $Q = 0$. To determine this speed we evaluate the double root condition $Q = \partial Q/\partial \mu = 0$, i.e.,
\[ \gamma^4 - \mu^2 - c\mu - 1 + iv = 0 \quad \text{and} \quad 4\gamma^3 \mu - 2\mu - c = 0, \tag{15} \]
which is a necessary condition for the minimal front speed to exist. To find $c$ and $v$, we eliminate the eigenvalue $\mu$ using the conditions (24). We use the Sylvester’s Dialytic Method of Elimination (see A) to obtain
\[ R(c, v; \gamma) = 0, \tag{16} \]
where $R(\cdot)$ is the determinant of Sylvester’s matrix. Some of the determinant elements are complex. Therefore, Eq. (16) gives two real equations. These two equations are
\[ v(9\gamma c^2 + 16\gamma^2(\nu^2 - 3) - 16\gamma - 1) = 0, \tag{17} \]
\[ -27\gamma c^4 + 4(36\gamma + 1)c^2 + 128\gamma(6\gamma + 1)v^2 - 16(4\gamma + 1)^2 = 0. \tag{18} \]

Now we solve (17) and (18) for $c$ and $v$. When $v = 0$, Eq. (17) is already satisfied. Then from (18) (with $v = 0$), fortunately we obtain two solutions explicitly for the wave speed. These two wave speeds are
\[ c_1 = \frac{1 + 36\gamma - (1 - 12\gamma)^{3/2}}{54\gamma} \quad , \quad v_1 = 0, \tag{19} \]
\[ c_2 = \frac{1 + 36\gamma + (1 - 12\gamma)^{3/2}}{54\gamma} \quad , \quad v_2 = 0 \tag{20} \]
provided that $\gamma < 1/12$. Now there are two solutions for the double root conditions (17) and (18), the two speeds $c_1$ and $c_2$ with $v = 0$. These two speeds are plotted versus the parameter $\gamma$ and shown in Fig. 3(a), as solid lines (they meet at $\gamma = 1/12$). A third possible solution of (17) and (18) for $v \neq 0$ exists, and can be determined explicitly as
\[ c_3 = \frac{-17 - 72\gamma - 2(7 + 24\gamma)^{3/2}}{54\gamma} \quad , \tag{21} \]
\[ v_3 = \frac{37 + 129\gamma + 144\gamma^2 - 2(7 + 24\gamma)^{3/2}}{54\gamma} \tag{22} \]
provided that $\gamma \geq 1/12$. The variation of $c_1$ and $v_3$ with $\gamma$ is shown in Fig. 3(a) and (b), as dashed lines.

Now we have all possible solutions of (17) and (18), which represent the wave speed and the modulating frequency that meet the double eigenvalue condition. We need to give insights on the character of four eigenvalues: the double eigenvalue and the associated two roots of the characteristic equation (14). When $v = 0$, the coefficient sequence of the characteristic polynomial is $\gamma$, $0$, $-1$, $-c$, $-1$. Hence, there is only one sign change in the coefficients signs, and according to Descartes’ Rule of signs (see appendices in see appendices in [15]), there will be at most one real positive root. Also, one can apply Routh–Hurwitz (RH) criterion (see Appendices in see appendices in [15]). We find that the RH conditions are not satisfied (some of the characteristic polynomial coefficients are negative), hence there is at least one root with positive real part. From this result, Descartes’ and RH criteria, we can say that for $v = 0$, there always a positive root of the characteristic equation. The other three roots are: one negative and two which are either negative or complex with negative real part. Fig. 3(c) shows the four eigenvalues when $v = 0$ and $\gamma = 0.02 < 1/12$ versus wave speed $c$. Two negative double roots exist, one at speed $c = c_1$ and the other at $c_2$, $c_1 < c_2$. A positive real root always exists, and the other three roots are negative, that occurs when $c_1 \leq c \leq c_2$, and otherwise they are one negative and the other two are complex with negative real part. At $c_1$ the double root is negative and the other negative root is decaying faster. Hence for $c = c_1$ only one exponential must be excluded as $z \to \infty$ for the repeated root to dominate there and this is the speed selected in practice for the EFK equation. However, at $c_2$ both the other exponential must be excluded as the double root is negative and the other two are one negative which is decaying slower than the double eigenvalue and one is positive.

From Fig. 3(a) that when $\gamma = \gamma_c = 1/12$ the three wave speeds $c_1$, $c_2$ and $c_3$ coincide and at this point $v = 0$. It is obvious from Fig. 3(b) that a transition from zero to nonzero modulating frequency $v$ takes place when $\gamma = 1/12$, this occurs when a triple root exists. Also, we can see from Fig. 3(c), as $\gamma \to 1/12$ the two speeds $c_1$ and $c_2$ are very close, and when $\gamma = 1/12$ a negative real triple root arises. Therefore we can say that a transition occurs when the characteristic equation (14) has a real triple root at $v = 0$. We can determine the transition point by solving the triple root conditions
\[ Q = \partial Q/\partial \mu = \partial^2 Q/\partial \mu^2 = 0 \quad (v = 0), \tag{23} \]
where $Q$ is the characteristic equation which displayed in (14). Hence these conditions appear as
and when we solve these equations, we find $c = \frac{1}{12}$ and $m = \frac{1}{2}$ which are shown in Fig. 3(a) and (b) in dashed lines. At $c = c_3$ the characteristic equation (14) has a complex double root with negative real part, and the other two roots are complex, one with negative real part (decaying faster than the double root) and the other with positive real part, see Fig. 3(d).

Fig. 4 shows the character of a double root discussed above. The plots in this figure are constant speed $c$ and frequency $\nu$ contours in $Re\mu$, $Im\mu$ space. A saddle point indicates that a double root exists. Equations which represent these contours can be obtained by substituting $\mu = X + iY$ into the characteristic equation (14) and then simplifying to obtain (see the caption of Fig. 4)

$$\gamma \mu^4 - \mu^2 - c\mu - 1 = 0, \quad 4\gamma \mu^3 - 2\mu - c = 0, \quad 6\gamma \mu^2 - 1 = 0$$

and when $\gamma > 1/12$, the double root speed and the corresponding modulating frequency $\nu$ are displayed in (21) and (22), which are shown in Fig. 3(a) and (b) in dashed lines. At $c = c_3$ the characteristic equation (14) has a complex double root with negative real part, and the other two roots are complex, one with negative real part (decaying faster than the double root) and the other with positive real part, see Fig. 3(d).

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$$\gamma (X^4 - 6X^2Y^2 + Y^4) - X^2 + Y^2 - cX - 1 = 0,$$
$$Y(\gamma [-2X^4 - 2X^2Y^2 + Y^4] + X^2 + Y^2 - 1) - \nu X = 0.$$
To justify the above results, we solve the EFK equation (10) when $c = 0$, $0.02 < 1/12$, at $t = 20$, 30 and 40 (see Fig. 5. We observe a uniform translating front travels to the right with a minimal speed $c = c_1$ displayed in (19). Solutions when $c = 1$, $0.02 > 1/12$ at $t = 20$, 30 and 40, are shown in Fig. 6. There is a pattern left behind the front invading the unstable state $u = 0$ with a minimal linear speed $c = c_1$ shown in (21). These solutions are constructed using Mathematica 8 Package (NDSolve).

3.2. Swift–Hohenberg equation

Here we give insights into propagating properties in the Swift–Hohenberg (SH) equation. By using the double root mechanism we show that the SH equation

$$
\frac{\partial u}{\partial t} = (\epsilon - 1)u - 2\frac{\partial^2 u}{\partial x^2} - \frac{\partial^4 u}{\partial x^4} - u^3, \quad 0 < \epsilon < 1,
$$

(27)
supports pattern formed front solutions. There is one steady state $u = u_c = 0$, and in the following we discuss its instability type. At $u = 0$, the dispersion relation of (27) (assuming that the perturbation is in the form $e^{i k x} e^{\sigma t}$) is

![Fig. 4. Re$\mu$, Im$\mu$ space plot. Constant speed contours, Eq. (25): (a) $\gamma = 0.05$, two real double roots exist at $\nu = 0$ and speeds $c_1 = 1.94$ and $c_2 = 2.13$, (b) $\gamma = 1/12$, triple root exits at $\nu = 0$ and $c = c_1 = 1.89$ and (c) $\gamma = 1$, a complex double root exits at speed $c_1 = 2.49$ and $v_1 = 0.76$. (d) Re$\mu$, Im$\mu$ versus $\nu$, Eq. (26), at $\gamma = 1$, a double root exists at the saddle point (when $\nu = 0.76$).]
\[
\frac{1}{2}k^2 + \frac{1}{C_0^2}k^4 + (\varepsilon - 1): \tag{28}\]

Thus for \(0 < \varepsilon < 1\), there is a band of wave numbers \(k_- < k < k_+\), for which \(u = 0\) is unstable (the fastest growing mode \(k = 1\)), see Fig. 7. Therefore, we can say that a pattern can arise as a result of disturbing the zero state.
To discuss the propagation properties of the SH equation solution, we linearize equation (27) around \( u = 0 \) (unstable steady state) and then in the traveling wave coordinates \((z, t) = (x - ct, t)\) the perturbation equation is

\[
\frac{\partial u}{\partial t} - c \frac{\partial u}{\partial z} = -2 \frac{\partial^2 u}{\partial z^2} - \frac{\partial^4 u}{\partial z^4} + (\varepsilon - 1)u,
\]

where \( u = u(z, t) = u(x - ct, t) \), assuming that \( u \) is the perturbation (here we dropped hats), and \( c \) is the wave speed. Then we substitute \( u = u_0 e^{i\mu t} e^{i\nu z} \) to obtain the following characteristic equation

\[
\mu^4 + 2\mu^2 - c\mu - (\varepsilon - 1) + iv = 0,
\]

where the eigenvalue \( \mu \) is complex, while \( c \) and \( \nu \) are real.

Now we find the double root condition. As in the previous section, the quartic equation \( Q = \mu^4 + 2\mu^2 - c\mu - (\varepsilon - 1) + iv = 0 \) results in two real equations, which can be put in the form

\[
27c^4 + 32c^2(9\varepsilon - 8) + 256(\nu^2 + \varepsilon^3 - 3\nu^2\varepsilon) = 0,
\]

\[
\nu(8(\nu^2 - 3\varepsilon^2 + 2\varepsilon) - 9c^2) = 0
\]

Fig. 8. (a) and (b) Double root locus for the SH equation, (a) speed versus \( \varepsilon \), (b) modulating frequency versus \( \varepsilon \), solid line represents (33) where a positive double root, and on the dashed lines a double complex root with negative real part, from (34) and (35). (c) The four roots versus the wave speed, solid lines represent the real part and dotted lines for imaginary parts, a positive double root exits at \( c = c_1 \) and the other two roots are complex with negative real parts (the roots of (30) when \( \nu = 0 \) and \( \varepsilon = 0.3 \)). (d) The real parts of a double root (dashed line) and the corresponding other two roots (solid lines, one positive and the other is negative and less than the double root) versus \( \varepsilon \), the roots computed at \( c = c_2 \) and \( \nu = \nu_2 \) shown in (34) and (35).
there is a complex double root with negative real part. Hence, in practice, a real positive double root exists, is the speed selected in practice and one with positive real part, the double root is the dominant, see Fig. 8(d).

The four eigenvalues versus the speed for the SH equation. Thus we can say that a front invades the unstable state \( u = 0 \), the front moving to the right. These solution are obtained using Mathematica 8 Package NDSolve.

and when we solve for \( c \) and \( v \), we find two possible solutions. The first is given by

\[
c_1 = \frac{4}{3\sqrt{3}} \left[ 8 - 9\varepsilon + (4 - 3\varepsilon)^{3/2} \right]^{1/2}, \quad v_1 = 0,
\]

which corresponds to a positive double eigenvalue for (30) (we will see later when we discuss the roots’ character), while the other solution is

\[
c_2 = \frac{4}{3\sqrt{3}} \left[ -1 + 18\varepsilon + (1 + 6\varepsilon)^{3/2} \right]^{1/2},
\]

\[
v_2 = \frac{4}{\sqrt{3}} \left[ -2 + 30\varepsilon + 9\varepsilon^2 + 2(1 + 6\varepsilon)^{3/2} \right]^{1/2},
\]

where a double complex root with negative real part exists. Fig. 8(a) and (b) show these two solutions versus the parameter \( \varepsilon \) (0 < \( \varepsilon \) < 1).

For the eigenvalues’ character, when \( v = 0 \). From (30), and when 0 < \( \varepsilon \) < 1, the coefficients’ signs always support two sign changes. According to Descartes’ rule, there will be at most two positive real roots or none. Since not all the polynomial coefficients are negative, the RH criterion is not satisfied. Therefore, there will be at least one complex root with positive real part. From this result, we can say that the characteristic equation must have two roots, which are either positive or complex with positive real parts (a positive double root exists at \( c = c_1 \)), and the other two roots are complex with negative real part. The four eigenvalues versus the speed \( m \) are shown in Fig. 8(c), the roots of (30) when \( v = 0 \) and \( \varepsilon = 0.3 \). For a nonzero value for \( v \), Eq. (30) always has a complex double root with negative real part, and the other two roots are complex, one with negative and one with positive real part, the double root is the dominant, see Fig. 8(d).

From the above investigation of the eigenvalues, we can deduce the following. At \( c = c_1 \) a real positive double root exists, while at \( c = c_2 \) there is a complex double root with negative real part. Hence, in practice, \( c = c_2 \) is the speed selected in practice for the SH equation. Thus we can say that a front invades the unstable state \( u = 0 \) with a minimal speed \( c = c_2 \), and leaves a pattern behind. Fig. 9 shows solutions of the SH equation when \( \varepsilon = 0.1 \) at successive times. Therefore, Eq. (27) supports patterned front solutions.

4. Summary

We have discussed the propagation properties in two examples of higher order reaction–diffusion equations, the EFK and the SH equation. We have demonstrated how to recognize two types of front solutions, uniform translating and patterned fronts for these two equations. A linear front speed is determined using the double root condition. In the EFK equation a transition from uniform translating font to patterned front is recognized and we have proved that happened when the characteristic equation has a triple root, similar results for the EFK equation obtained by the marginal stability mechanism (see, [23,4,20]). The double root speed and the associated angular frequency are computed from the resultant of the characteristic equation using the Sylvester’s Dialytic Method of Elimination. To ensure the existence of a front, we need to give insight on the character of the double roots and the other two roots (with the help of RH criterion and Descartes’ rule of signs, Appendix A, as it is hard to obtain the four roots explicitly). The double root has to be slowest decaying one present, for the selection mechanism to make sense.

This work gives a brief outline on the properties of a front solution of two examples of reaction–diffusion equation of fourth order, and we consider the analysis as a motivation and a guide when reaction–diffusion systems are considered as a future work.

Fig. 9. Solution of the SH equation when \( \varepsilon = 0.1 \), and at \( t = 70, 90, 110 \) (left to right). The initial condition is a Gaussian of height 0.01. A pattern invades the unstable state \( u = 0 \), the front moving to the right. These solution are obtained using Mathematica 8 Package NDSolve.
Acknowledgments

I acknowledge both Prof. John King and Dr. Stephen Cox (School of Mathematical Sciences, University of Nottingham, UK), for their help and useful advice.

Appendix A. Sylvester’s Method of Elimination

If \( p \) and \( q \) are two polynomials which can be factored into linear factors
\[
\begin{align*}
\text{of } p(x) &= a_0(x - r_1)(x - r_2) \cdots (x - r_m), \\
\text{of } q(x) &= b_0(x - s_1)(x - s_2) \cdots (x - s_n),
\end{align*}
\]
then the resultant \( R(f, g) \) of \( f \) and \( g \) is defined as
\[
R_s(p, q) = a_0^n b_0^m \prod_{i=1}^{m} \prod_{j=1}^{n} (r_i - s_j).
\]

From the definition, it is clear that the resultant will equal zero if and only if \( p \) and \( q \) have at least one common root. An explicit formula for the resultant as a determinant was given by Sylvester [1]. Suppose that
\[
\begin{align*}
\text{of } p(x) &= a_0 x^m + a_1 x^{m-1} + \cdots + a_{m-1} x + a_m, \\
\text{of } q(x) &= b_0 x^n + b_1 x^{n-1} + \cdots + b_{n-1} x + b_n.
\end{align*}
\]
Then \( R_s(p, q) \) can be expressed as an \( (m + n) \times (m + n) \) determinant:
\[
\begin{bmatrix}
a_0 & a_1 & a_2 & \cdots & a_m & 0 & \cdots & 0 \\
0 & a_1 & a_2 & \cdots & a_m & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & 0 & \cdots & a_m & 0 & \cdots & 0 \\
b_0 & b_1 & b_2 & \cdots & b_n & 0 & \cdots & 0 \\
0 & b_0 & b_1 & \cdots & b_{n-1} & b_n & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & b_n \\
\end{bmatrix}
= R_s(p, q).
\]

To construct this determinant, one first lists the coefficients of \( p \), padded with zeros at the end, then constructs subsequent rows by shifting one column to the right each time until one runs out of zeros at the end, then one repeats the same procedure with \( q \). Resultants are very useful for solving simultaneous systems of polynomial equations. The resultant allows one to eliminate a variable from a system of equations. For this reason, resultants are also known as eliminant. By using resultants to eliminate variables repeatedly one variable at a time, one solve systems of equations in more than two unknowns.

Consider the case in which \( q(x) \) is the derivative of \( p(x) \) with respect to \( x \). In this case, there must be at least one common root; a double root of \( p(x) \) exists. Hence we can say that resultant \( R_s(p, q) \) must be zero, i.e.,
\[
R_s \left( p, \frac{dp}{dx} \right) = 0.
\]

Consider a cubic polynomial equation
\[
p(x) = A_3 \mu^3 + A_2 \mu^2 + A_1 \mu + A_0 = 0,
\]
then we construct the resultant using (A.6) and apply (A.7) to obtain
\[
\begin{bmatrix}
A_3 & A_2 & A_1 & A_0 & 0 \\
0 & A_3 & A_2 & A_1 & A_0 \\
3A_3 & 2A_2 & A_1 & 0 & 0 \\
0 & 3A_3 & 2A_2 & A_1 & 0 \\
0 & 0 & 3A_3 & 2A_2 & A_1 \\
\end{bmatrix}
= 0,
\]
which can be simplified to
\[
4A_3 A_1^3 - A_2^2 A_1^2 - 18A_2 A_3 A_4 + 4A_2^3 + 27A_3^2 = 0,
\]
which is a condition for \( p(x) \) to have a double root. Also, a condition for a double root of the quartic polynomial equation
\[ A_4 \mu^4 + A_3 \mu^3 + A_2 \mu^2 + A_1 \mu + A_0 = 0 \]  \hspace{1cm} (A.11)

can be obtained. This condition is
\[
\begin{array}{cccccc}
A_4 & A_3 & A_2 & A_1 & A_0 & 0 \\
0 & A_4 & A_3 & A_2 & A_1 & A_0 \\
0 & 0 & A_4 & A_3 & A_2 & A_1 & A_0 \\
4A_4 & 3A_3 & 2A_2 & A_1 & 0 & 0 \\
0 & 4A_4 & 3A_3 & 2A_2 & A_1 & 0 \\
0 & 0 & 4A_4 & 3A_3 & 2A_2 & A_1 \\
0 & 0 & 0 & 4A_4 & 3A_3 & 2A_2 & A_1 \\
\end{array} = 0 \hspace{1cm} (A.12)
\]

and can appear as
\[
-27A_0^2A_4^2 + 18A_2A_4A_2A_4^2 - 4A_2A_4A_4^2 + 144A_0A_2A_4A_2^2 - 4A_2A_4A_4^2 - 6A_0A_2A_4A_2^2 - 192A_0A_4A_2A_4^2 - 27A_0^2A_4^2 \\
- 80A_0A_2A_4A_2A_4 - 18A_4A_2A_4A_2A_4 + 256A_4A_2^2 - 128A_0A_2A_4^2 + 16A_0A_2A_4^2 + 144A_0A_2A_4A_2A_4 - 27A_0^2A_4A_2A_4 \\
- 4A_0A_2A_4A_2A_4 = 0. \hspace{1cm} (A.13)
\]

References